

# Experimental identification for inverse problem of a mechanical system with a non-minimum phase based on singular value decomposition

Sung-Jong Kim and Sang-Kwon Lee\*

*Department of Mechanical Engineering, Inha University, 253 Yonghyun Dong, Incheon, 402-751, Korea*

(Manuscript Received January 31, 2008; Revised March 20, 2008; Accepted March 20, 2008)

---

## Abstract

This paper presents experimental results of source identification for a non-minimum phase system. Generally, a causal linear system may be described by matrix form. The inverse problem is considered as a matrix inversion. Direct inverse method cannot be applied for a non-minimum phase system, because the system has ill-conditioning. Therefore, in this study the SVD inverse technique is introduced to execute an effective inversion. In a non-minimum phase system, its system matrix may be singular or near-singular and has very small singular values. These very small singular values have information about a phase of the system and ill-conditioning. Using this property, we could solve the ill-conditioned problem of the system and then verify it for the practical system (cantilever beam). The experimental results show that the SVD inverse technique works well for a non-minimum phase system. This inverse technique can be applied to the estimation of the magnitude of impact force, which becomes often a cause of damage to a mechanical system.

*Keywords:* Singular value decomposition; Non-minimum phase; Inverse ill-conditioning matrix

---

## 1. Introduction

The inverse problem is one of the important topics in engineering research [1] and several methods have been introduced for its analysis. Cepstrum analysis has been used for the inverse problem in time domain [2, 3]. However, this method is not suitable for the inverse problem of a causal linear system with non-minimum phase. Matrix inversion has been considered as a solution of the inverse problem [4]. Direct inverse of a matrix also cannot be applied for a non-minimum phase system because the system has ill-conditioning. Therefore, in this study the SVD inverse technique is introduced to execute an effective inversion. In a non-minimum phase system, its system matrix may be singular or near-singular and have small singular values. These very small singular values have information about the phase of the system

and ill-conditioning. Using this property, we could solve the ill-conditioned problem of the system and then verify it with the practical system (cantilever beam). The experimental results show that the SVD inverse technique works well for a non-minimum phase system. This inverse technique can be applied to the estimation of the magnitude of impact force, which often becomes the cause of damage to a mechanical system.

## 2. Input and output of a system

The output of a linear causal system is given by the convolution between input and impulse response of a system. Its mathematical expression is given by

$$y(n) = h(n) * x(n) + b(n) \quad (1)$$

where  $y(n)$  is the output of a system,  $h(n)$  is an impulse response of a system,  $x(n)$  is input of system and  $b(n)$  is a noise component. The convolution in Eq. (1) is again expressed by [4]

---

\*Corresponding author. Tel.: +82 32 860 7305, Fax.: +82 32 868 1716  
E-mail address: sangkwon@inha.ac.kr  
© KSME & Springer 2008

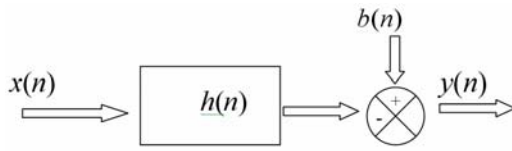


Fig. 1. Input-output relationship.

$$y(n) = \sum_{k=0}^n x(k)h(n-k) + b(n) \tag{2}$$

This convolution is rewritten as a matrix form as follows:

$$\begin{bmatrix} y(0) \\ y(1) \\ \dots \\ y(m) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & \dots & 0 \\ h(1) & h(0) & \dots & 0 \\ \dots & \dots & \dots & 0 \\ h(m) & h(m-1) & \dots & h(m-n) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \dots \\ x(n) \end{bmatrix} + \begin{bmatrix} b(0) \\ b(1) \\ \dots \\ b(n) \end{bmatrix} \tag{3}$$

or

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{4}$$

where  $\mathbf{y}$  is an output vector,  $\mathbf{x}$  is an input vector,  $\mathbf{A}$  is system matrix and  $\mathbf{b}$  is a noise vector. This relationship is graphically shown in Fig. 1.

### 3. Inverse transform of a non-minimum phase

Transfer function  $H(z)$  of a system is obtained by Z-transforming the impulse response  $h(n)$  of a system. The zero of transfer function  $H(z)$  of a system with minimum phase is located inside of the unit circle as shown in Fig. 2(a), while that of a system with non-minimum phase is located outside of the unit circle as shown in Fig. 2(c). Although the singular value of the system matrix  $\mathbf{A}$  of a system with minimum phase does not have a smallest value as shown in Fig. 2(b), the singular value of a system matrix  $\mathbf{A}$  of a system with non-minimum phase has very small singular values. The number of smallest singular values is the same as the number of the zeroes located outside of the unit circle as shown in Fig. 2(d). There are two smallest singular values in Fig. 2(d). For a system with a minimum phase, the input of the system can be directly obtained by matrix inversion since the zeroes of the transfer function  $H(z)$  become poles, and poles will be located inside of the unit circle after inversion.

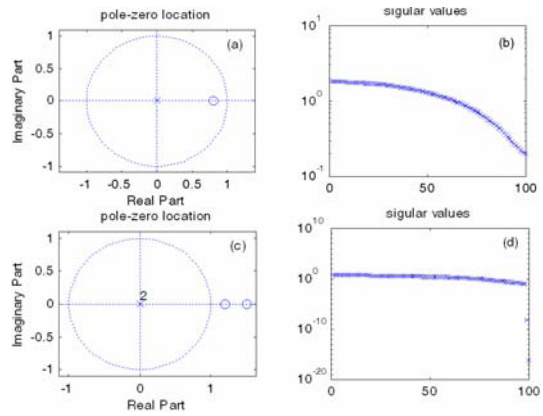


Fig. 2. Distribution of singular values of the minimum phase system and non-minimum phase system.

Therefore, the inversion of a system is stable. The input of a system is thus expressed by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{5}$$

For a system with a non-minimum phase, the input of the system cannot be obtained by direct matrix inversion of system matrix  $\mathbf{A}$  since the poles of the system are located outside of the unit circle after inversion. Therefore, the inversion of the system is unstable and the system matrix has ill-conditioning [6, 7]. In this case, the input of the system is obtained by singular value decomposition method and its solution is given by,

$$\hat{\mathbf{x}} = \hat{\mathbf{A}}^{-1}\hat{\mathbf{y}} \tag{6}$$

where  $\hat{\mathbf{A}} = \mathbf{U}\hat{\mathbf{S}}\mathbf{V}^T$  and  $\hat{\mathbf{S}}$  is obtained by changing the diagonal matrix  $\mathbf{S}$ . The diagonal elements of diagonal matrix  $\mathbf{S}$  consist of singular values of the system matrix  $\mathbf{A}$  as shown in Fig. 2(d). New diagonal matrix  $\hat{\mathbf{S}}$  is obtained by replacing the smallest singular values of the matrix  $\mathbf{S}$  with a reasonable value compared with other singular values of the diagonal matrix  $\mathbf{S}$ . On the other hand, the system matrix  $\mathbf{A}$  is  $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$  and  $\mathbf{U}$  and  $\mathbf{V}$  are rectangular matrices.  $\hat{\mathbf{y}}$  is the output of a system without noise. Noise component  $b(n)$  is removed by using an SVD (singular value decomposition) filter [7].  $\hat{\mathbf{x}}$  is the recovered system input.

## 4. Experiment identification

### 4.1 Set-up of experimental equipment

The method developed for the inverse problem is applied to the estimation of input force exciting a

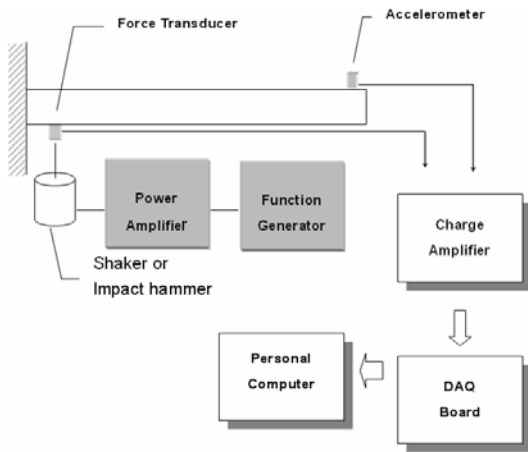


Fig. 3. Setup for experimental equipment.

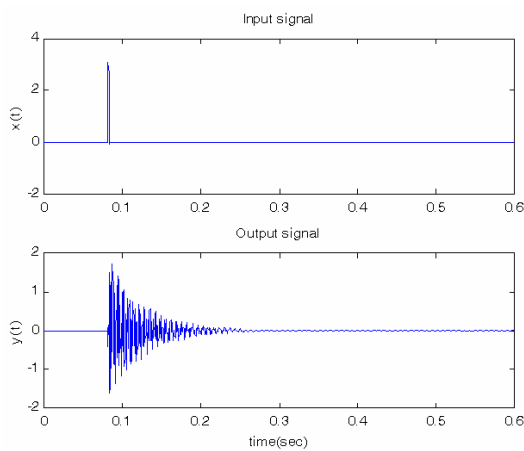


Fig. 4. Input forces  $x(n)$  and impulse response  $y(n)$ .

cantilever beam with non-minimum phase. The experimental equipment is arranged as shown in Fig. 3. The cantilever beam for the test is welded in strong structure. An accelerometer is installed on the end of the cantilever beam. The beam is excited at the other end, as shown in Fig. 3, by two excitation methods: impact hammer and electric shaker. The measured force and vibration signals are amplified and are transferred to a personal computer through an A/D converter.

**4.2 Impulse response of a beam**

The element of system matrix  $\mathbf{A}$  of a beam is obtained from the impulse response of the beam by using Eqs. (3) and (4). As a first method, the impulse response is measured by exciting the beam using the

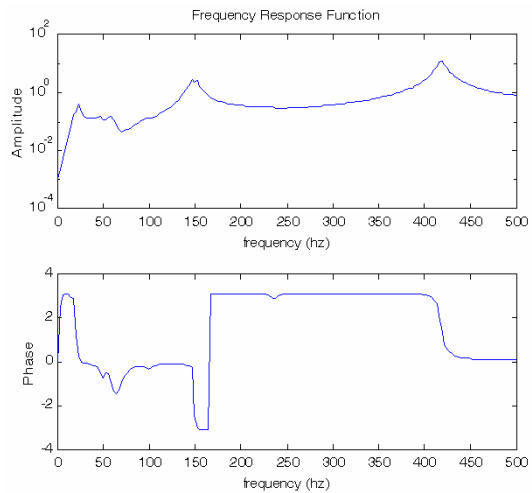


Fig. 5. Magnitude and phase of the estimated frequency response of a system.

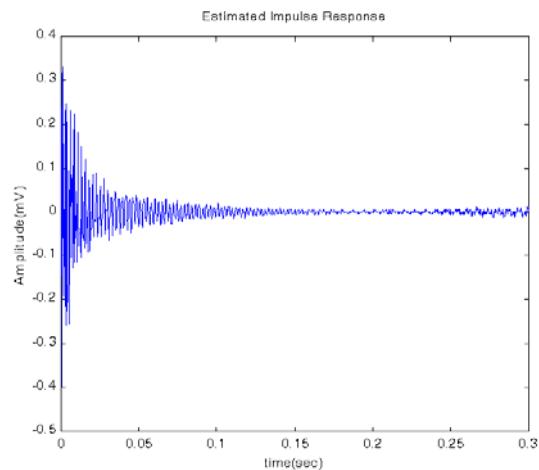


Fig. 6. Estimated impulse response of a system.

impact hammer. Fig. 4(a) shows the impact signal  $x(t)$  measured through a force transducer of the impact hammer. Fig. 4(b) shows the impulse response  $h(n)$  measured through an accelerometer installed on the beam. The second method is to use an electric shaker. The input force  $x(t)$  used for the excitation of a beam is random. The output of a system  $y(n)$  is measured with an accelerometer. The frequency response function  $H(f)$  is calculated by using Eq. (7) as follows [8]:

$$\hat{H}(f) = \frac{\hat{G}_{xy}(f)}{\hat{G}_{xx}(f)} \tag{7}$$

where  $G_{xy}(f)$  is the estimation of cross spectrum between input  $x(t)$  and output  $y(t)$ , and  $G_{xx}(f)$  is the

estimation of auto spectrum of input signal  $x(t)$ . Fig. 5 shows the estimation of frequency response  $\hat{H}(f)$ . The impulse response  $h(n)$  of a system using an electric shaker is calculated by taking the inverse Fourier transform of the frequency response function  $\hat{H}(f)$ . Fig. 6 shows the estimated impulse response of a beam using an electric shaker.

**4.3 Poles and zeroes of the transfer function**

The characteristics of a system are associated with the location of poles and zeroes. The system transfer function  $H(z)$  is calculated the Z-transform of the impulse response  $h(n)$ . The poles and zeroes are plotted as shown in Fig. 7. Most of the zeroes are located inside of the unit circle, but four zeroes are located outside of the unit circle. Therefore, this system is a non-minimum phase system. For this system, the input force  $x(t)$  cannot be calculated by direct inversion of a system matrix  $\mathbf{A}$  using Eq. (5) because the system is non-minimum phase. Therefore, the SVD inverse technique with Eq. (6) has been used for the estimation of input force. A flow chart of the SVD inverse technique is shown in Fig. 8. Noise component  $b(n)$  is removed by using the SVD filter after matrix inversion.

**4.4 Results and discussions**

Two impact forces and one sinusoidal force are given to excite the cantilever beam. These input forces should be recovered by SVD inverse technique. Fig. 9 and Fig. 11 show the impact input forces  $x(n)$  and its responses  $y(n)$ . First, one has only one impact force and the second one has double impacts. In order to recover impact forces, the direct matrix inverse with Eq. (5) is used and its results are plotted in Figs. 10(a) and 12(a). This method is not suitable for estimation of impact force since the system is a non-minimum phase system. The SVD inverse technique using Eq. (6) is applied, and its results are shown in Figs. 10(b) and 12(b). The impact forces are clearly recovered without any phase change but with little noise component. A sinusoidal input force  $x(n)$  using an electric shaker is also applied to the cantilever beam. Fig. 13(a) and Fig. 14(a) show the sinusoidal input force  $x(n)$ . In order to recover this force, two impulse responses as shown in Fig. 4 and Fig. 6 are used. Input force recovered by using Fig. 4 is plotted in Fig. 13 and the input force recovered by using Fig. 6 is plotted in Fig. 14. At the first, the direct matrix

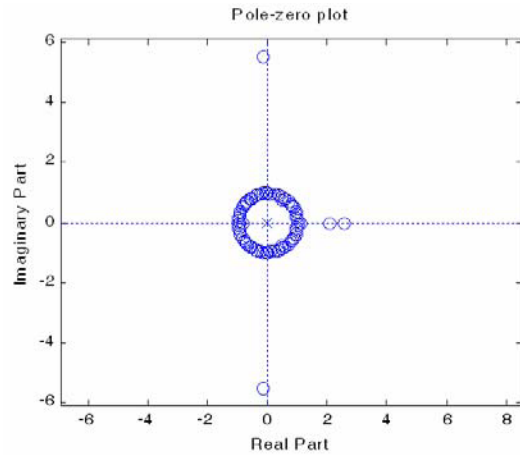


Fig. 7. Pole-zero plot of z-transform of the finite impulse response.

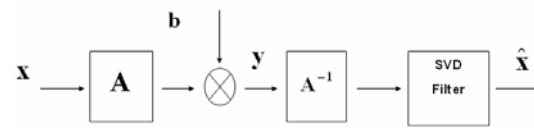


Fig. 8. Flow chart for SVD inverse technique.

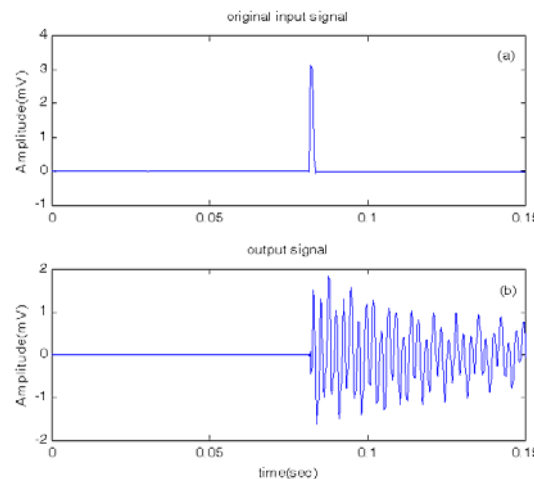


Fig. 9. Single impact force and response (a) original input (impact) signal, (b) output (response) signal.

inversion method with Eq. (5) was used, and its results are plotted in Figs. 13(b) and 14(b). It was not successful in recovering the force. The SVD inverse technique was applied and its results are plotted in Fig. 13(c) and Fig. 14(c). The sinusoidal force recovered with the impulse response  $h(n)$  obtained by the impact hammer has a phase difference with the original input force as shown in Fig. 13(a) and Fig. 13(c). The

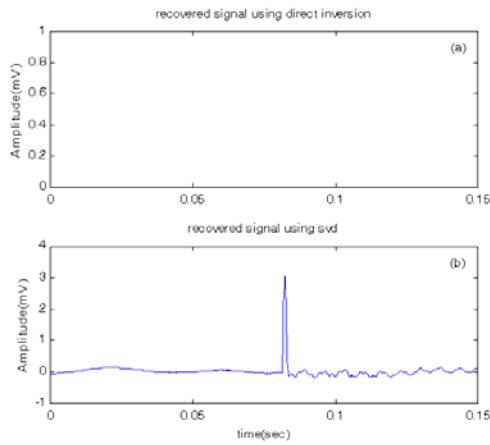


Fig. 10. Recovering of an impact source (a) direct method, (b) recovered signal using SVD inverse technique.

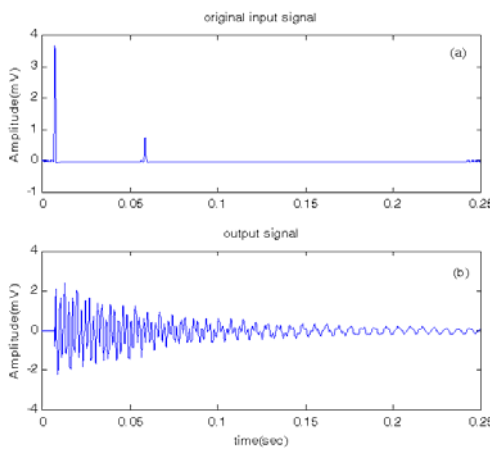


Fig. 11. Double impact force and response (a) original input (impact) signal, (b) output (response) signal.

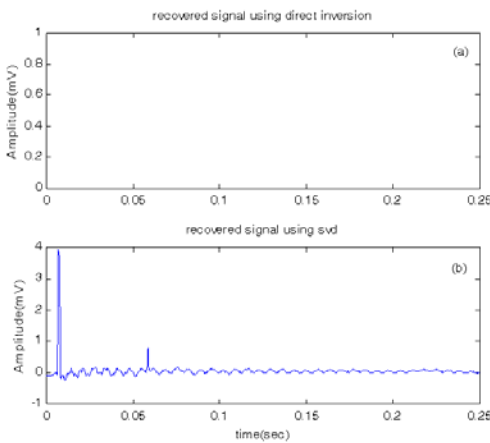


Fig. 12. Recovering an impact source (a) direct method (b) recovered signal using SVD inverse technique.

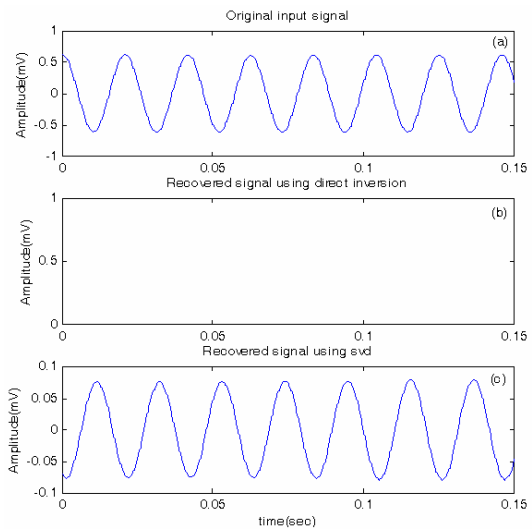


Fig. 13. Recovering a sinusoidal source using impulse response function obtained by a impact hammer (a) Original input signal (sine signal), (b) recovered signal using direct method, (c) recovered signal using SVD inverse technique.

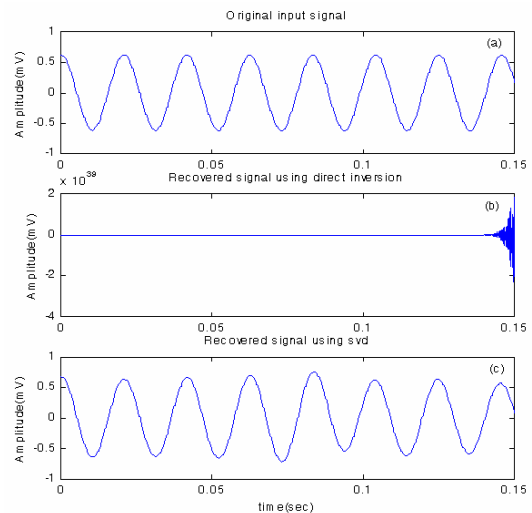


Fig. 14. Recovering a signal using impulse response function obtained by a shaker (a) Original input signal (sine signal), (b) recovered signal using direct method, (c) recovered signal using SVD inverse technique.

sinusoidal force recovered with the impulse response  $h(n)$  obtained by the shaker is nearly the same as the original sinusoidal input force as shown in Fig. 14(a) and Fig. 14(c) with only small amplitude change. This difference is due to the window effect used when calculating the impulse force using Eq. (7).

## 5. Conclusion

The SVD inverse technique was developed to estimate the input force of a system with non-minimum phase. This method was successfully applied to the estimation of input force exciting a cantilever beam with non-minimum phase. The impulse response of a system is important for estimating the input force. The direct matrix inversion method is not suitable for recovering the input force exciting a system with non-minimum phase.

## References

- [1] M. Bertero and C. D. Mol, SVD for Linear Inverse Problems, Elsevier, (1995).
- [2] R. H. Lyon, Machinery Noise and Diagnostics, Butterworths, London, (1986).
- [3] J. T. Kim and R. H. Lyon Cepstral Analysis As a Tool for Robust Processing, Deverberation and Detection of Transients, *Mechanical Systems and Signal Processing*, 6 (1992) 1-15.
- [4] S. Haykin, Adaptive Filter Theory, Prentice Hall, 2<sup>nd</sup> Edition, (1986).
- [5] S. Hashemi and J. K. Hammond, The Interpretation of Singular Values in the Inversion of Minimum and Non-minimum phase systems, *Mechanical Systems and Signal Processing*, 10 (1996) 225-240.
- [6] G. H. Golub and C. F. Van Loan, Matrix Computation, John Hopkins University Press, (1996).
- [7] P. K. Sadasivan and D. Narayana Dutt, SVD based technique for Noise Reduction in Electroencephalographic Signal, *Signal Processing*, 55 (1996) 179-189.
- [8] J. S. Bendat and A. G. Piersol, Random data: Analysis and Measurement Procedures, John-Wiley & Sons, (2000).